2023-24 MATH2048: Honours Linear Algebra II

Homework 5 Answer

Due: 2023-10-16 (Monday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date. All questions are selected from Friedberg §2.4-2.5.

1. For each of the following pairs of ordered bases β and β' for $P_2(R)$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

   (a) $\beta = \{x^2, x, 1\}$ and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$

   (b) $\beta = \{1, x, x^2\}$ and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$

   Solution.

   (a) For $\beta = \{x^2, x, 1\}$ and $\beta' = \{v_1 = a_2x^2 + a_1x + a_0, v_2 = b_2x^2 + b_1x + b_0, v_3 = c_2x^2 + c_1x + c_0\}$, the change of coordinate matrix that changes β'-coordinates into β-coordinates is given by:

   $$[[v_1]_\beta, [v_2]_\beta, [v_3]_\beta] = \begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{bmatrix}$$

   (b) Similarly, the change of coordinate matrix that changes β'-coordinates into β-coordinates is also given by:

   $$\begin{bmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

2. For each matrix A and ordered basis β, find $[L_A]_\beta$. Also, find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$.

(a) $A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

(b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

Solution.

(a) Note that $L_A(e_1) = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6e_1 - 2e_2$ and $L_A(e_2) = \begin{pmatrix} 7 \\ 3 \end{pmatrix} = 11e_1 - 4e_2$. Therefore, the matrix representation with respect to β is

$$[L_A]_\beta = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}.$$

The matrix Q is the change of basis matrix from β to the standard basis, i.e.,

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

It can be easily checked that this matrix indeed satisfies $[L_A]_\beta = Q^{-1}AQ$.

(b) Again, $L_A(e_1) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3e_1$ and $L_A(e_2) = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -e_2$. Therefore, the matrix is

$$[L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix},$$

and the matrix Q is just

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

3. Let $T : V \to W$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let β and γ be ordered bases for V and W, respectively. Prove that $\text{rank}(T) = \text{rank}(L_A)$ and that $\text{nullity}(T) = \text{nullity}(L_A)$, where $A = [T]_\beta^\gamma$. Hint: Apply Exercise 17 to Figure 2.2.

Proof. We want to show that $\text{rank}(T) = \text{rank}(L_A)$ and that $\text{nullity}(T) = \text{nullity}(L_A)$. The following commutative diagram illustrates the relationship between T and L_A:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \downarrow \phi_\beta & & \downarrow \phi_\gamma \\ \mathbb{R}^n & \xrightarrow{L_A} & \mathbb{R}^m \end{array}$$

where $\phi_\beta = [-]_\beta$ and $\phi_\gamma = [-]_\gamma$ are the coordinate maps associated with the bases β and γ, and are linear isomorphisms.

By the diagram, $L_A \circ \phi_\beta = \phi_\gamma \circ T$. Then $L_A(\mathbb{R}^n) = L_A \circ \phi_\beta(V) = \phi_\gamma \circ T(V)$. Since ϕ_γ is a linear isomorphism, $\dim(T(V)) = \dim(\phi_\gamma(T(V)))$. Therefore, $\text{rank}(T) = \dim(T(V)) = \dim(\phi_\gamma(T(V))) = \dim(L_A(\mathbb{R}^n)) = \text{rank}(L_A)$.

Similar reasoning applies to the null spaces. Since ϕ_γ is an isomorphism, $\phi_\gamma^{-1}(0) = 0$. Then $\ker(T) = T^{-1}(0) = T^{-1}\phi_\gamma^{-1}(0) = (\phi_\gamma \circ T)^{-1}(0) = (L_A \circ \phi_\beta)^{-1}(0) = \phi_\beta^{-1} \circ L_A^{-1}(0) = \phi_\beta^{-1}(\ker(L_A))$.

Since ϕ_β is an isomorphism, $\ker(T) = \phi_\beta^{-1}(\ker(L_A))$ has the same dimension as $\ker(L_A)$. Therefore, $\text{nullity}(T) = \text{nullity}(L_A)$. □

Remark. Since all dimensions are finite, we can prove either one equality, and get the other by the rank-nullity theorem (rank+nullity=source dim).

4. Let $c_0, c_1, \ldots, c_n$ be distinct scalars from an infinite field F. Define a transformation $T : P_n(F) \to F^{n+1}$ by $T(f) = (f(c_0), f(c_1), \ldots, f(c_n))$. Prove that T is an isomorphism. Hint: Use the Lagrange polynomials associated with $c_0, c_1, \ldots, c_n$.

Proof. Let $c_0, c_1, \ldots, c_n$ be distinct scalars from an infinite field F. We define a transformation $T : P_n(F) \to F^{n+1}$ by $T(f) = (f(c_0), f(c_1), \ldots, f(c_n))$.

To show that T is an isomorphism, we need to show that T is both injective and surjective. (While it would be sufficient to prove one of these properties and then apply the rank-nullity theorem, given that the dimensions of the source and target spaces agree and finite, we will explicitly show both properties for the sake of thoroughness and clarity.)

Injectivity: Suppose $T(f) = T(g)$ for some $f, g \in P_n(F)$. Then $f(c_i) = g(c_i)$ for all $i = 0, 1, \ldots, n$. Consider the polynomial $h(x) = f(x) - g(x)$. Then $h(c_i) = 0$ for all i. But $h(x)$ is a polynomial of degree at most n with $n+1$ roots, which can only be the zero polynomial. Therefore, $f = g$, so T is injective.

Surjectivity: Let $(a_0, a_1, \ldots, a_n) \in F^{n+1}$. We need to find a polynomial $f \in P_n(F)$ such that $T(f) = (a_0, a_1, \ldots, a_n)$. For this, we use the Lagrange polynomials. The Lagrange polynomial $L_i(x)$ associated with c_i is given by

$$L_i(x) = \prod_{j \neq i} \frac{x - c_j}{c_i - c_j}.$$

Then $L_i(c_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. We can then construct the polynomial $f(x) = \sum_{i=0}^{n} a_i L_i(x)$. For this polynomial, $f(c_i) = a_i$ for all i, so $T(f) = (a_0, a_1, \ldots, a_n)$. Therefore, T is surjective.

Since T is both injective and surjective, it is an isomorphism. □

5. Consider a linear transformation $T : V \to W$, where $\dim(V) = \dim(W) = n$. Show that if T has a left inverse U, then U is also a right inverse of T, thus T is invertible. (Hint. Sec. 2.4 Q10(b), prove it if you use it)

   Proof. Let $T : V \to W$ be a linear transformation, where $\dim(V) = \dim(W) = n$, and let $U : W \to V$ be a left inverse of T, i.e., $U \circ T = \mathrm{Id}_V$.

   Since $U \circ T = \mathrm{Id}_V$, we have that T is injective. Since $\dim(V) = \dim(W) = n$, by the rank-nullity theorem, the rank of T must be n, i.e., the image of T is the whole of V. Then T is bijective and has a set-theoretical right inverse U'.

   Then $TU' = \mathrm{Id} = UT$. Hence $U = UI_V = UTU' = I_V U' = U'$. Hence, $U = U'$ is a right inverse of T, and so T is invertible. □